Unexpected Curves Talk

Recall: The projective plane over $\mathbb{C}$.

$$\mathbb{P}^2_{\mathbb{C}}$$

$$\frac{\{(a,b,c)\in\mathbb{C}^3 : (a,b,c)\neq(0,0,0)\}}{(\lambda a,\lambda b,\lambda c)\sim(a,b,c)}$$

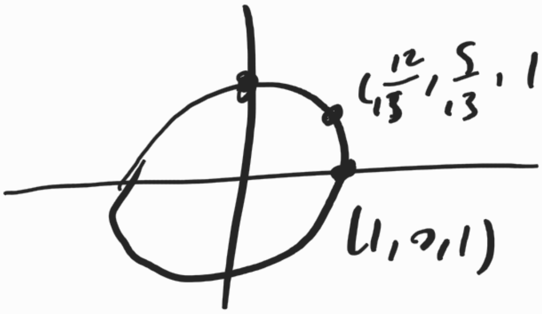
$\lambda\neq 0$.

Elements of $\mathbb{P}^2_{\mathbb{C}}$ can be thought of as a ratio of three numbers: $[a:b:c]$.

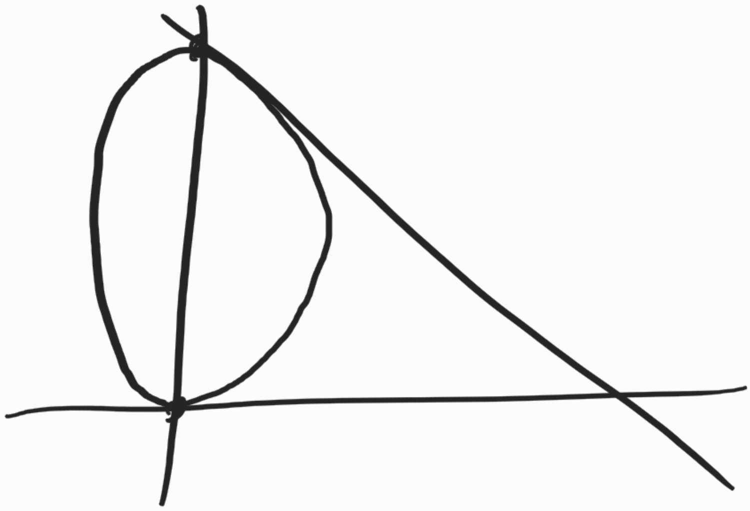
$[1:1:1]=[2:2:2]$.

Homogeneous polynomials define algebraic curves in $\mathbb{P}^2$.

ex. $Z(x^2+y^2-z^2)$.



$Z(y^2-xz)$.



Return to: how many points determines a unique curve of degree? Line: $3-1=2$.

$$Z\left(\begin{vmatrix} x & y & z \\ a & b & c \\ a' & b' & c' \end{vmatrix}\right)$$

Conic: 5

cubic: 9 (CB).

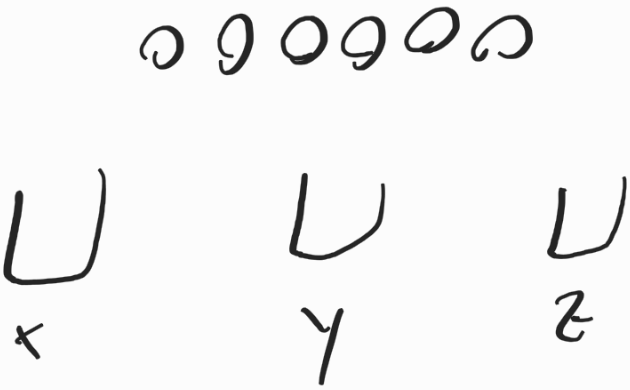
General d?

How many monomials of degree d can we form in three variables?

Rephrase: how many ways can we organize d balls into 3 bins?

Answer: $\binom{d+2}{2}$.

Example: 6 balls.



Encode each position with bits

tally works

8

$x^2 y^2 z^2$

$x^5 z$

Neat!

So: there are $\binom{d+2}{2}$ x, y, z monomials in degree d.

So: $\binom{d+2}{2} - 1$ points determines a degree d curve!

((C-B type weirdness notwithstanding))

Instead of using matrices most geometers see this as how many "conditions" we are imposing on a general degree-d polynomial.

Example: general conic

$$Ax^2 + Bxy + Cxz + Dy^2 + Eyz + Fz^2.$$

(capital letters are f[illegible]).

(make affine).

we want it to go

through $(0,0,1)$.

Then we have the condition

$$F = 0.$$

Choosing points isn't the

only thing that imposes conditions!

Choosing slopes at points also.

Say we wanted the curve to have a horizontal slope at $(0,0,1)$. i.e. the tangent line is $y=0$.

Then we get the condition

$$C=0.$$

or tangent line is $x+y=0$.

Then $C=E$.

Still counts as one condition.

This is also called choosing an "infinitely-near point".

Finally: singularities impose conditions!

$$Ax^3 + Bx^2y + Cxy^2 + Dy^3 + Ex^2 + Fxy$$
$$+ Gy^2 + Hx + Iy + J = 0$$

want double point at $(0,0)$

Then $H = I = J = 0$.

Hardware: "Short term 'behavior' (i.e. near $(0,0)$) is determined by the lowest-degree terms of the defining polynomial."

Linear: $Ex^2 + Fxy + Gy^2$

$$= \lambda\left(2E\,x - y\left(-F + \sqrt{F^2 - 4EG}\right)\right)\left(2E\,x - y\left(-F - \sqrt{F^2 - 4EG}\right)\right)$$

fudge factor whatever

line.

line.

So zoomed in @ $(0,0)$, the curve looks like 2 lines!

example

$$x^2 + 3xy + 2y^2$$

$$= (x+y)(y+2y)$$

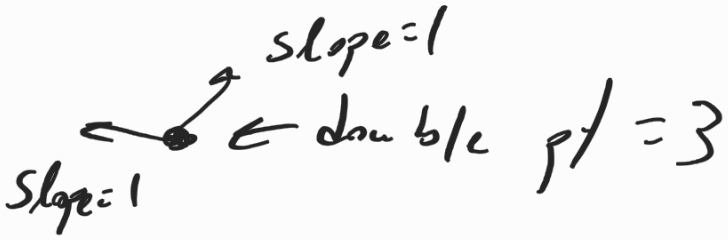
So in summary:

We get $\binom{d+2}{2}-1$ tokens.

- Points cost 1 token
- Slope costs 1 token
- Double point costs $\binom{2+1}{2}=3$ tokens.
- Triple point costs $\binom{3+1}{2}=6$ tokens.
- d-uple point costs $\binom{d+1}{2}$ tokens.

total

Example: conic



5 total.

Unexpected??

A degree d curve with a singularity of multiplicity d is just a union of d lines (not interesting).

But one of multiplicity $d-1$? Ok...

If we want a curve of degree d to have a $d-1$-singularity ...

we have $\binom{d+2}{2} - 1 - \binom{d}{2}$

$$= \frac{(d+2)(d+1)}{2} - \frac{d(d-1)}{2} - 1$$

$$= \frac{\cancel{d^2} + 3d + 2 - \cancel{d^2} + d - 2}{2}$$

$$= \frac{4d}{2} = 2d$$ tokens left to buy with.

So... asking for $2d+1$ points is too much.

I DO NOT ACCEPT THAT.

A configuration of $2d+1$ points admits an unexpected curve of degree d if...

there is a curve of degree d that goes through the points and also has a $d-1$-singularity wherever I want.

Unexpected cubic:

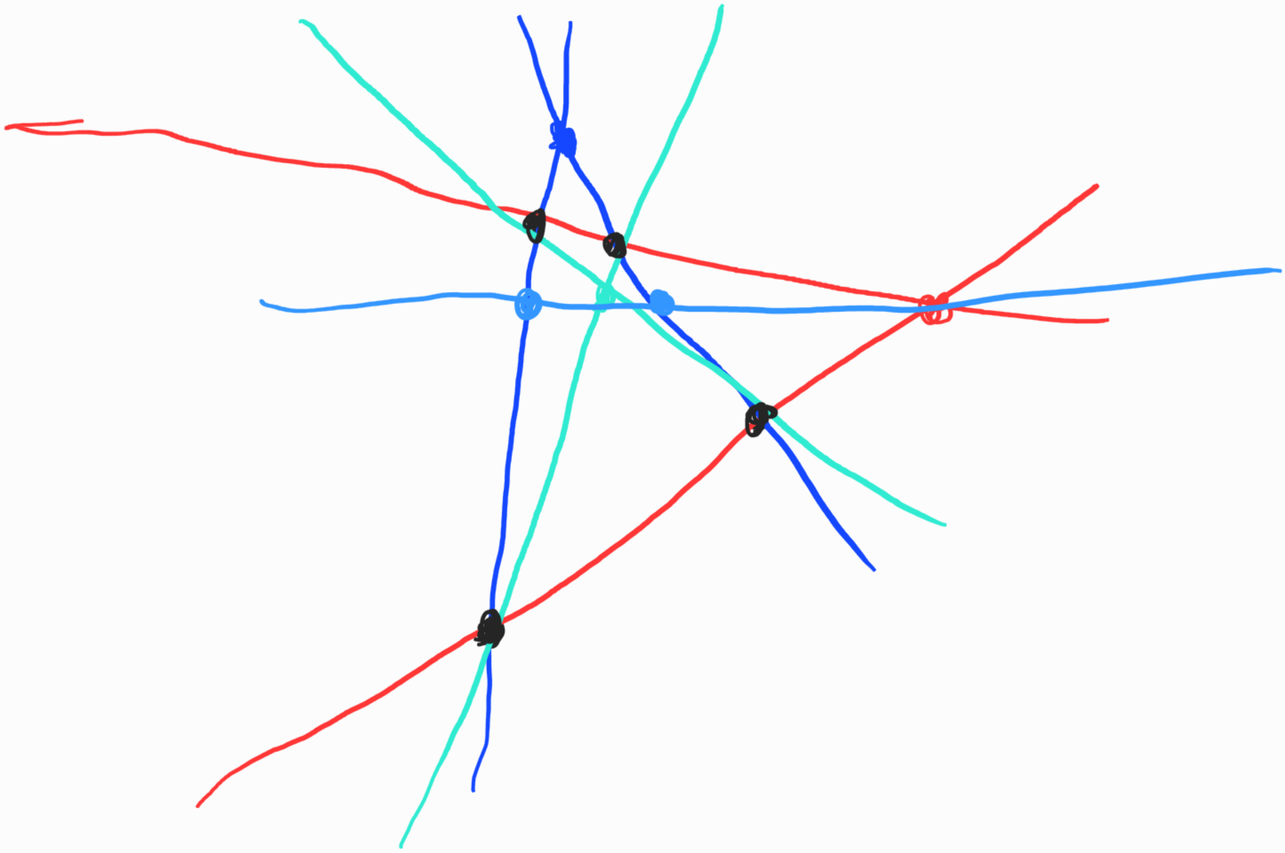
7 points & you can put a double pt wherever (but exist in characteristic $0, \geq 5$).

- Solomon looked at unexpected cubics in characteristic 2.
- Solomon proved that if you insist on picking 7 distinct points... there are no unexpected

cubics in characteristic 3.

In characteristic 0, the smallest degree unexpected curve is a quartic. Farnik-Galuppi-Sodomaco-Trok 2019

9 points w/ triple point wherever.



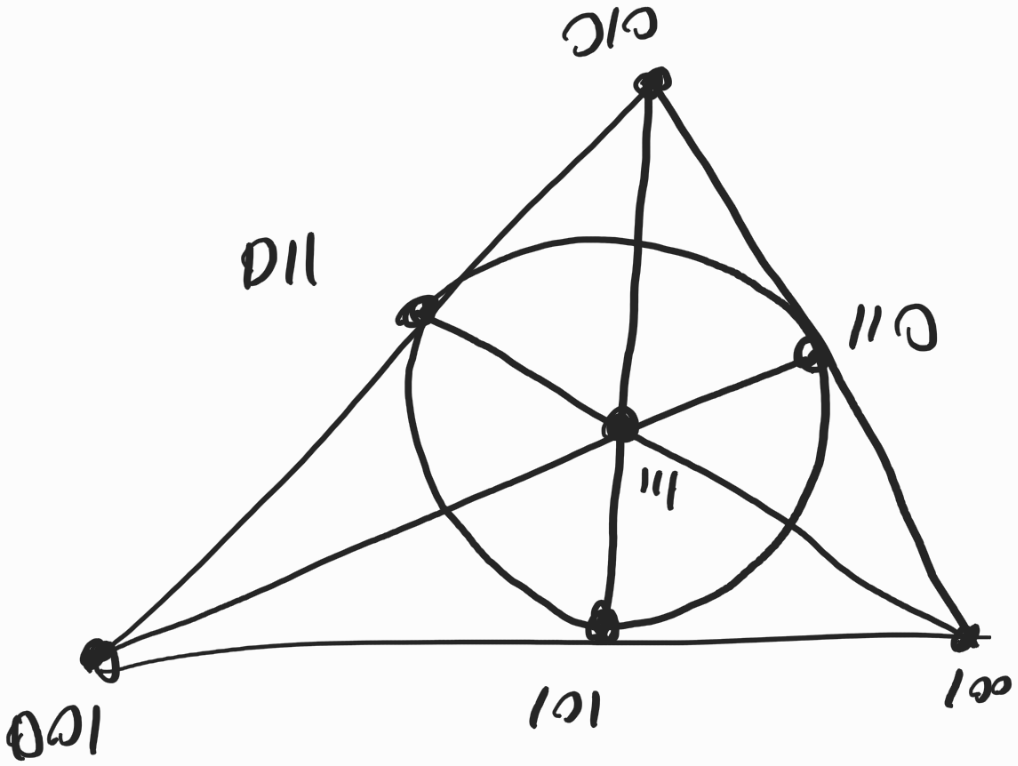
Only unexpected quartic configuration in characteristic 0.

In characteristic 2 … lots of unexpected cubics!

But only one with 7 distinct points.

(Akesseh)

7 points of $\mathbb{P}^2_{\mathbb{Z}/2\mathbb{Z}}$.



The double can be any point in $\mathbb{P}^2_{\mathbb{Z}/2\mathbb{Z}}$.